Windmills of the Minds

An Algorithm for Fermat's Two Squares Theorem

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Outline

1. Sum of Two Squares
2. Windmills
3. Algorithm

Sum of Two Squares

Primes

$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, \ldots$

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	7	
	11	
13		$13 = 3^2 + 2^2$
17		$17 = 1^2 + 4^2$
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If odd n is a sum of two squares, n = odd square + even square.

Two Squares Theorem

A tik prime is a sum of two squares, odd and even, uniquely.

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- 1801 Gauss revised Lagrange's proof, invented Gaussian integers.
- 1877, 1894 Dedekind offered 2 proofs, based on Gaussian integers.
- Many other proofs, based on:
  - Jacobi sums,
  - Pigeonhole principle,
  - Continued fractions,
  - ...

Windmills and Minds

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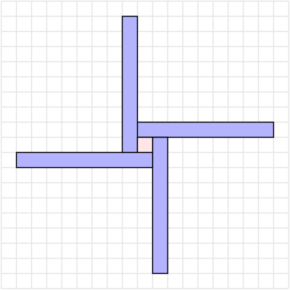

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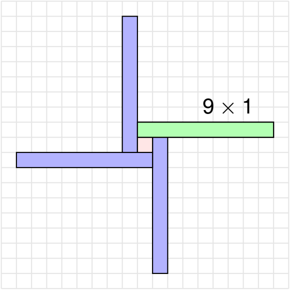

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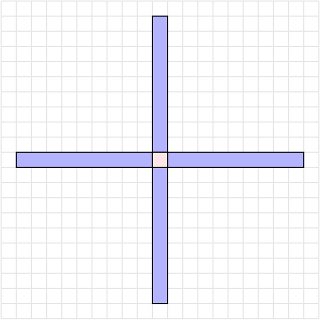

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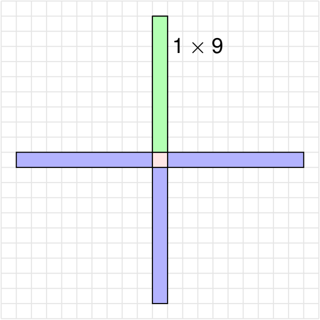

Another windmill!

Windmill

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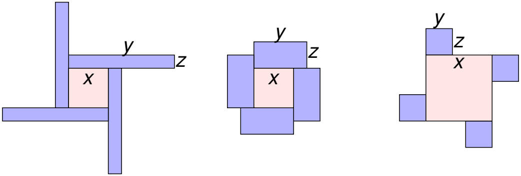
windmill $x\ y\ z \stackrel{\text{def}}{=} x^2 + 4yz$



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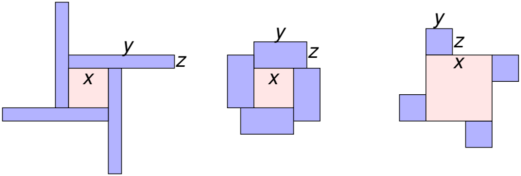

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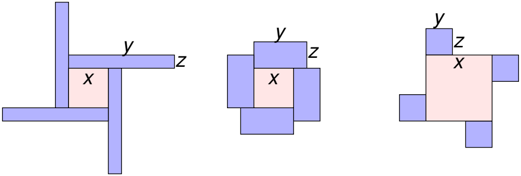

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- Every ✓ number has at least one windmill.

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3	$37 - 3^2$	$=$	$28 = 4(7)$	$(3, 1, 7), (3, 7, 1)$
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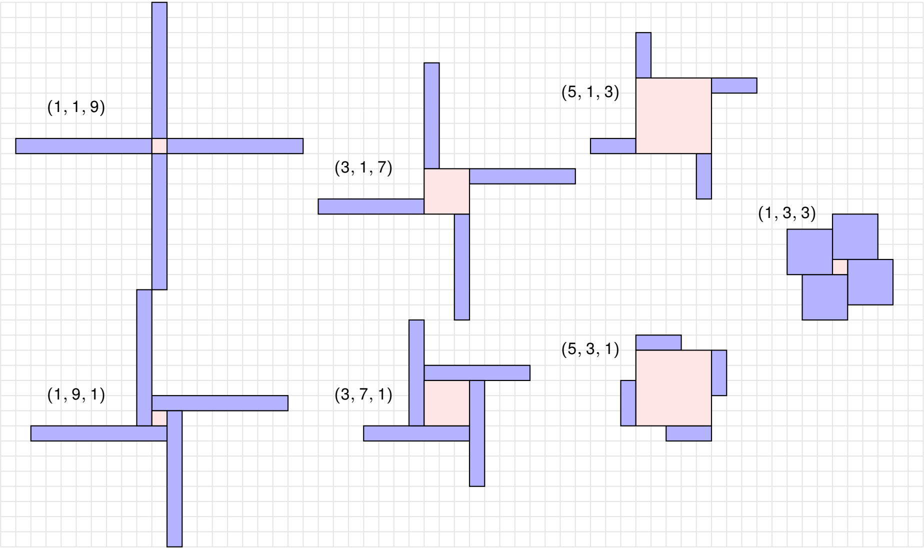
All the windmills of a ✓ number form its **mills.** Thus,

$$\text{mills } 37 = \{(1, 1, 9), (1, 9, 1), (1, 3, 3), (3, 1, 7), (3, 7, 1), (5, 1, 3), (5, 3, 1)\}$$

How many windmills? $|\text{mills } 37| = 7$, an odd number.

Windmills of $n = 37$



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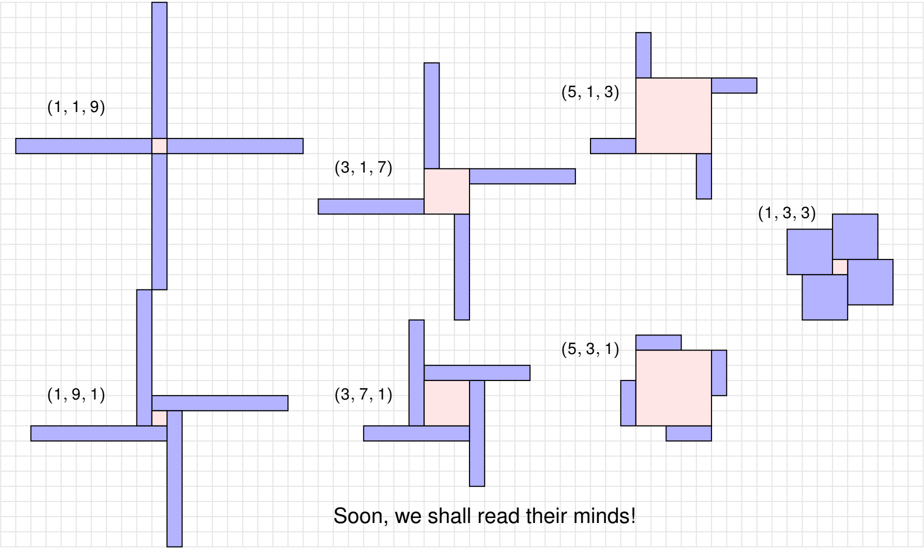
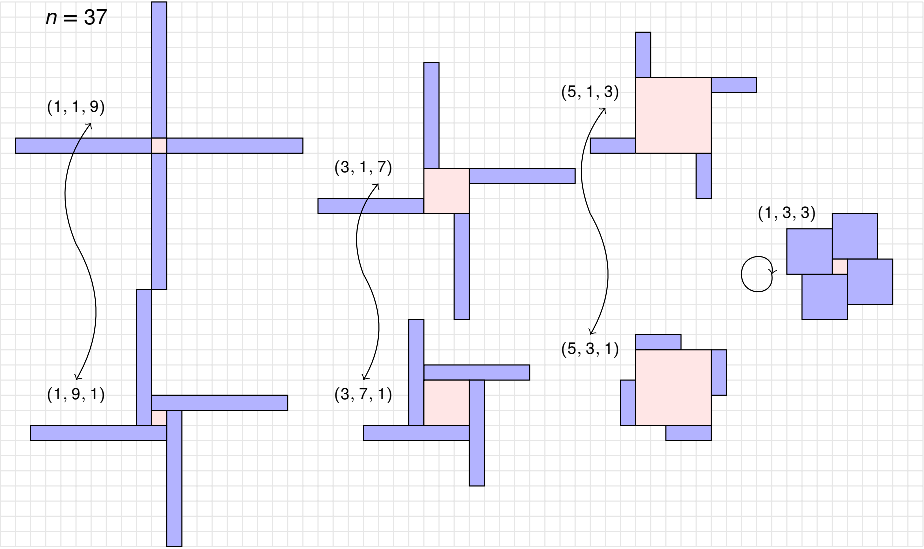


Soon, we shall read their minds!

Windmills related by Flips



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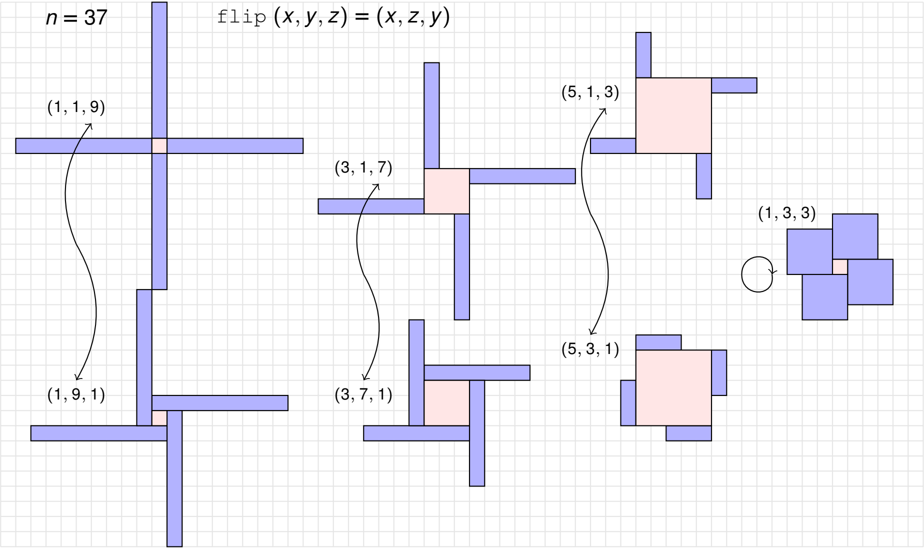

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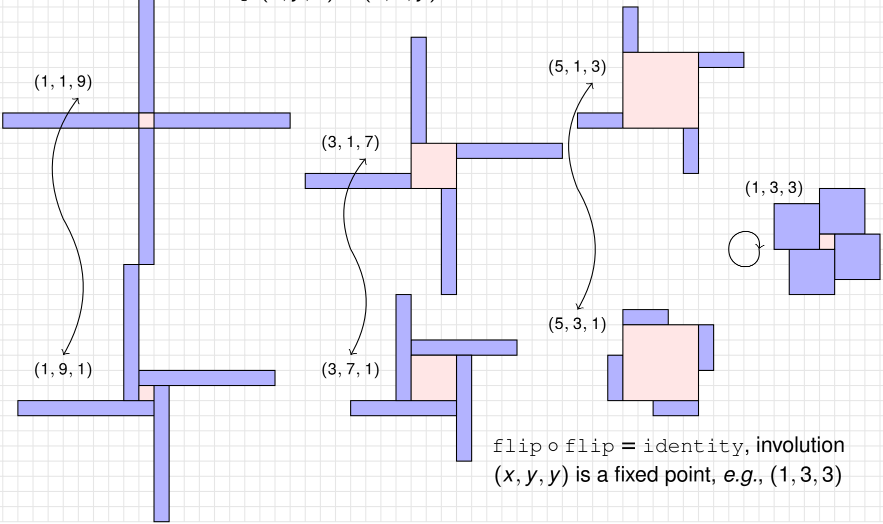
$n = 37$ `flip` $(x, y, z) = (x, z, y)$



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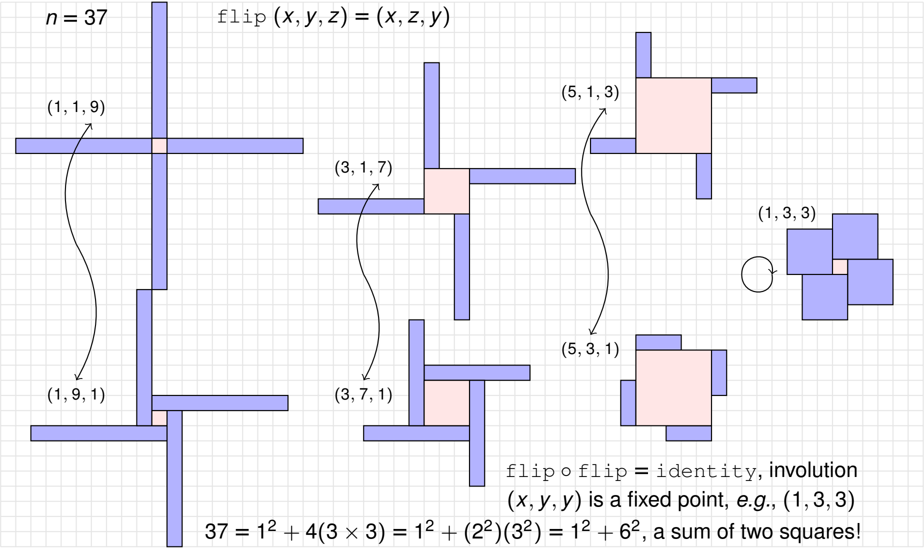

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$37 = 1^2 + 4(3 \times 3) = 1^2 + (2^2)(3^2) = 1^2 + 6^2$, a sum of two squares!

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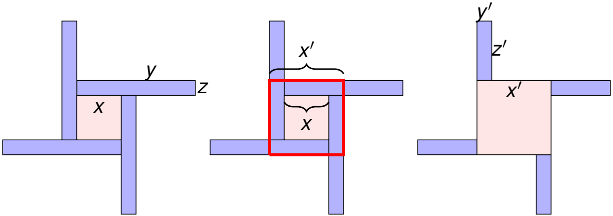

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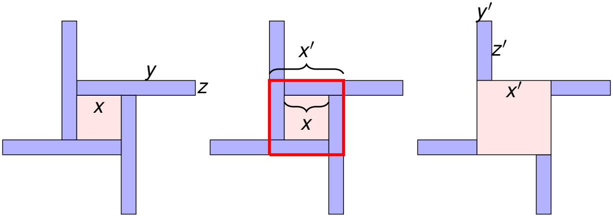

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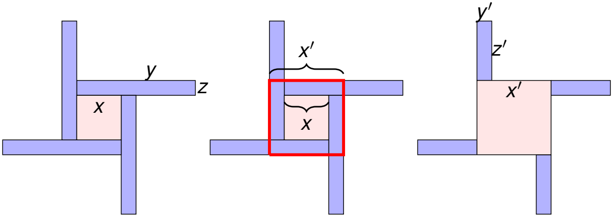

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$$\mathtt{mind}\,(x, y, z) = \text{length of square mind}$$

Using simple geometry,

$$\mathtt{mind}\,(x, y, z) = \begin{cases} x + 2z & \text{if } x < y - z \\ 2y - x & \text{else if } x < y \\ x & \text{otherwise} \end{cases}$$

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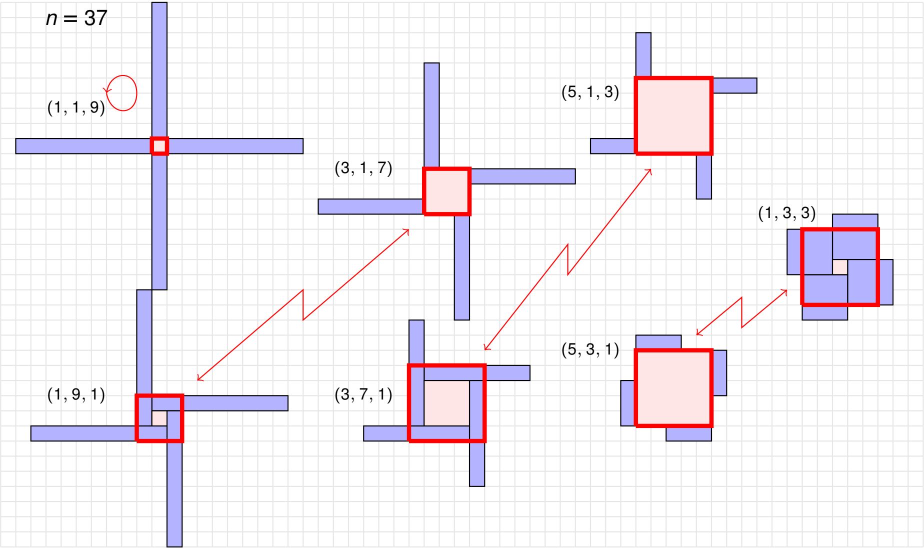

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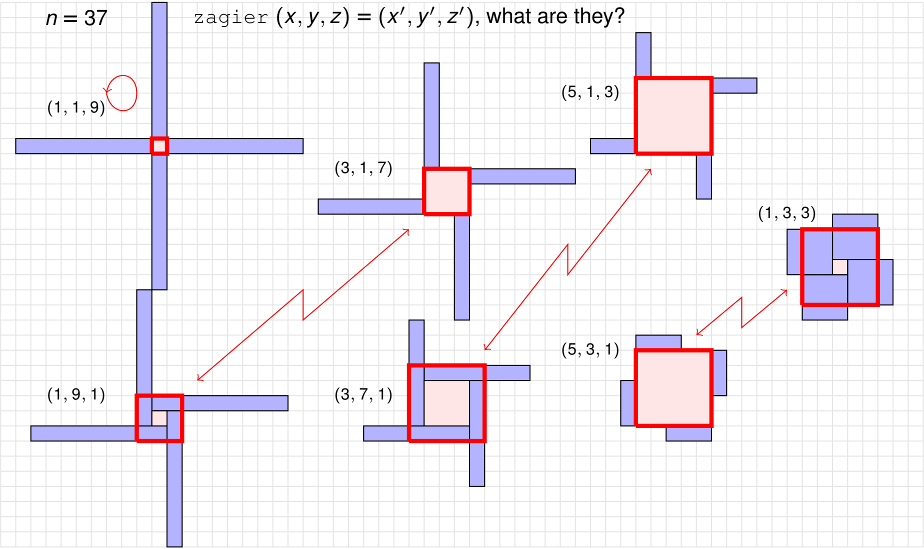
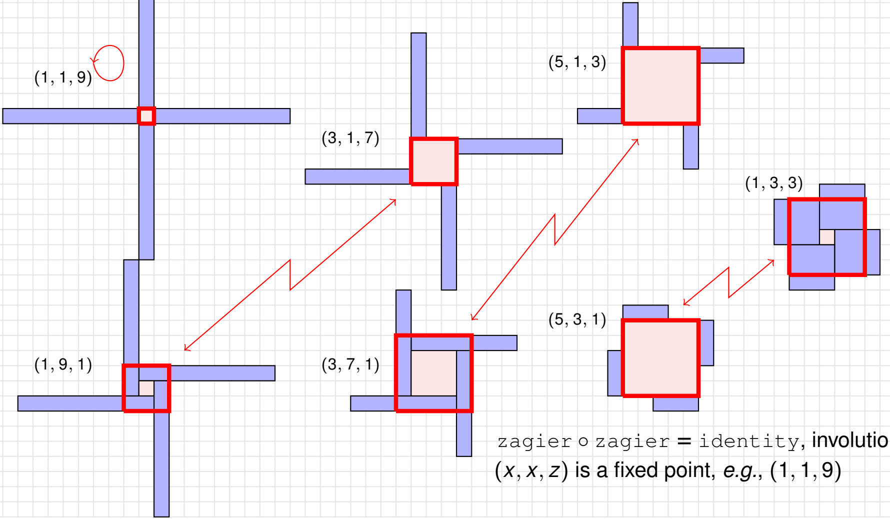


Windmills related by Minds

$n = 37$ `zagier` $(x, y, z) = (x', y', z')$, what are they?



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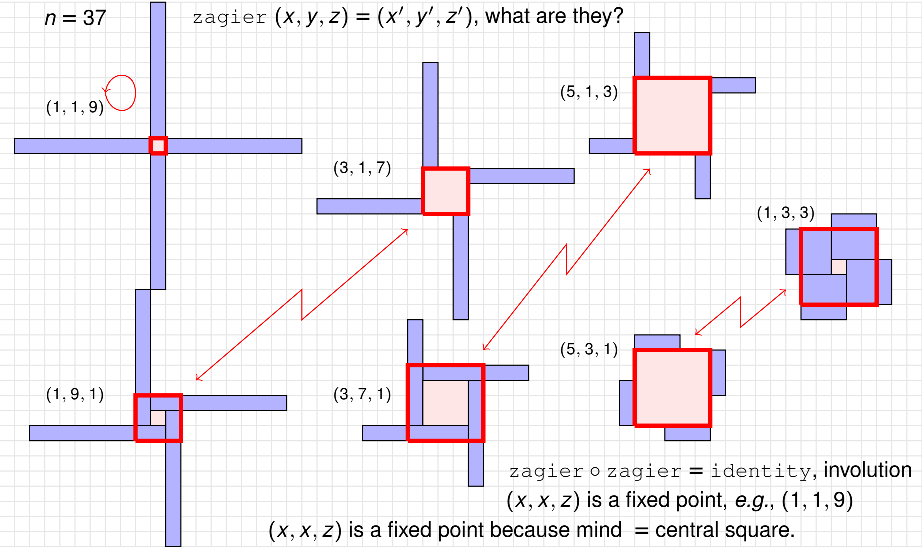

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(x, x, z) is a fixed point because mind = central square.

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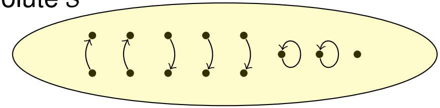

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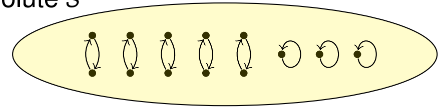

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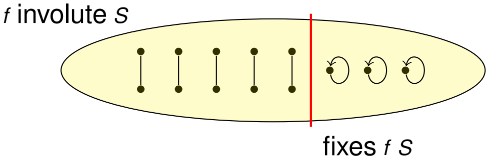

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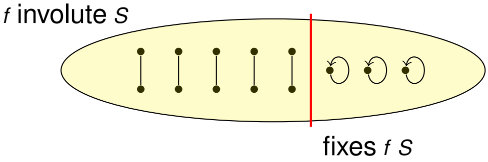

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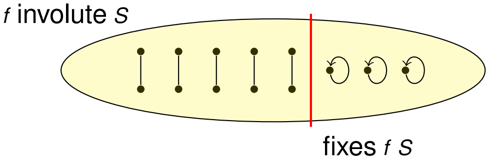

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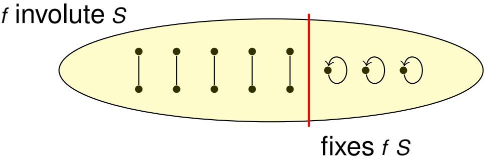

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- For any involution, size of set is *odd* $\Rightarrow$ *at least one* fixed point.
- Given a ✓prime $n = 4k + 1$, `zagier` fix is $(1, 1, k)$ only.
- So $|$mills $n|$ is **odd**, `flip` fix exists, n is a sum of two squares!

Zagier's Proof

A One-Sentence Proof That Every Prime $p \equiv 1 \pmod 4$ Is a Sum of Two Squares

D. ZAGIER
Department of Mathematics, University of Maryland, College Park, MD 20742

The involution on the finite set $S = \{(x, y, z) \in \mathbb{N}^3 : x^2 + 4yz = p\}$ defined by

$$(x, y, z) \mapsto \begin{cases} (x + 2z,\ z,\ y - x - z) & \text{if } x < y - z \\ (2y - x,\ y,\ x - y + z) & \text{if } y - z < x < 2y \\ (x - 2y,\ x - y + z,\ y) & \text{if } x > 2y \end{cases}$$

has exactly one fixed point, so $|S|$ is odd and the involution defined by $(x, y, z) \mapsto (x, z, y)$ also has a fixed point. □

Don Zagier's famous proof of Fermat's Two Squares Theorem, February 1990.

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All these have been formalised in other theorem provers!

Two Squares Algorithm

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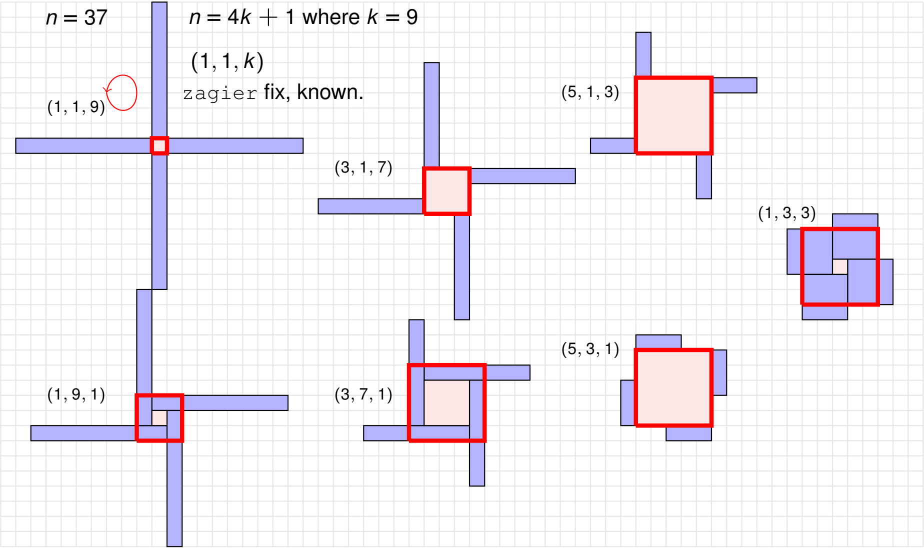


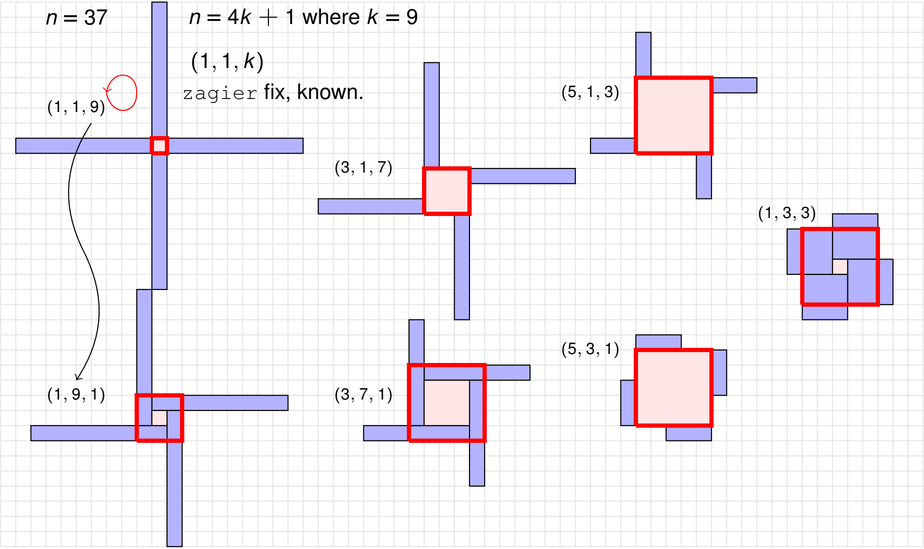
Two Squares Algorithm

$n = 37$

$n = 4k + 1$ where $k = 9$

$(1, 1, k)$

`zagier` fix, known.



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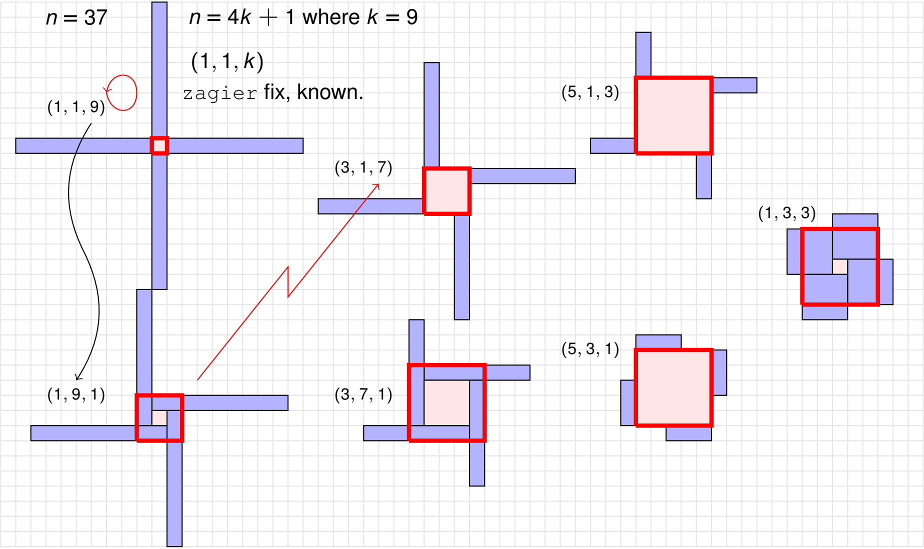

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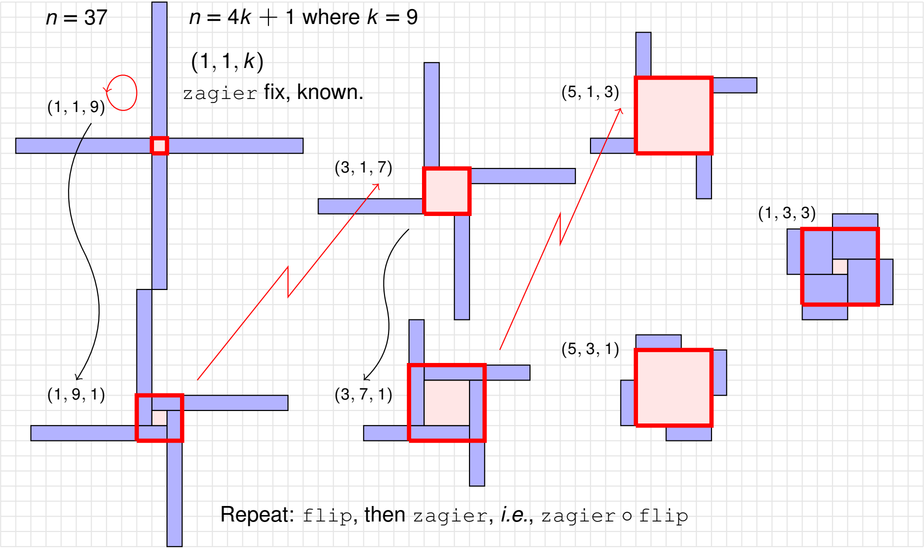
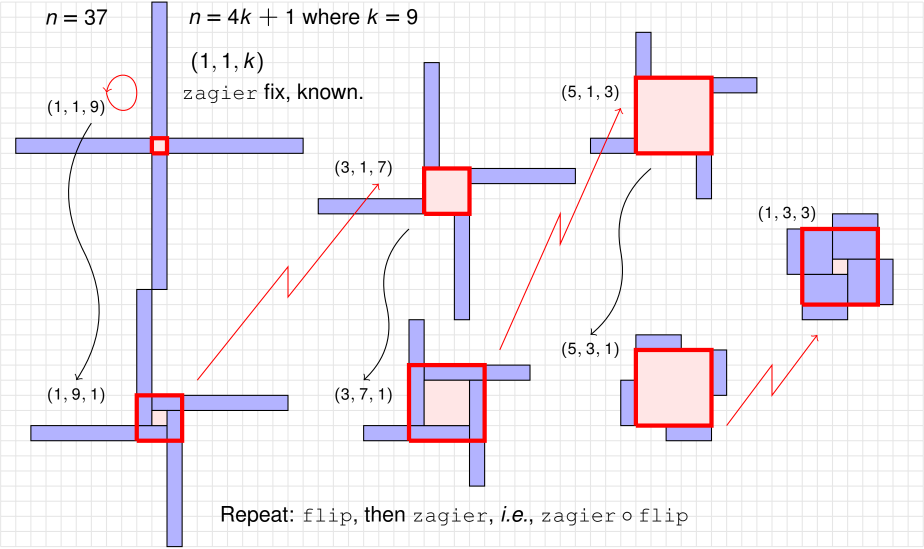


Repeat: `flip`, then `zagier`, *i.e.*, `zagier` $\circ$ `flip`

Two Squares Algorithm



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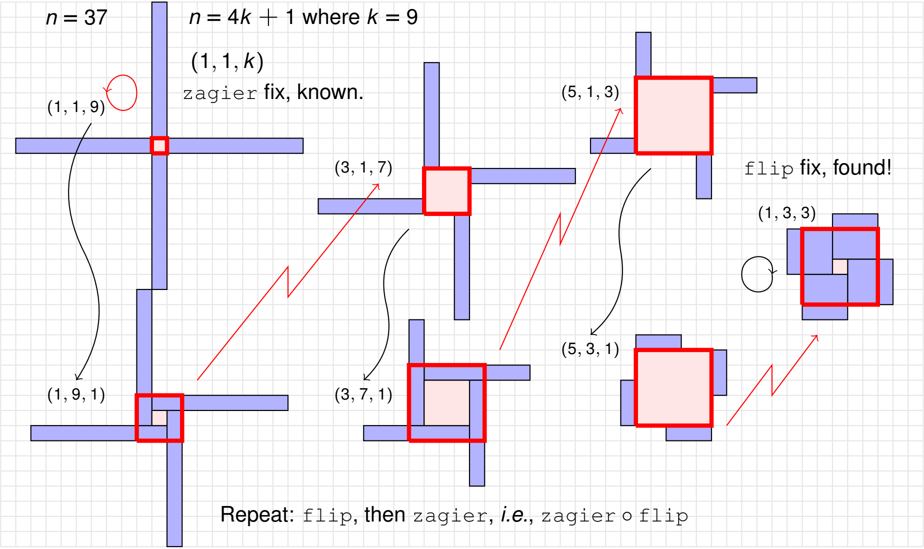

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Algorithm Code

- *Input*: a tik (✓) number $n = 4k + 1$.
- *Output*: a flip fix triple (x, y, y) so that $n = x^2 + (2y)^2$.

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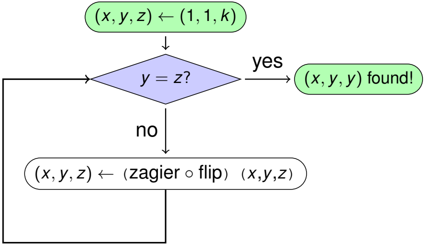

Method: use a triple t.

- start with $t = (1,1,k)$, the `zagier` fix.
- while (t is not of the form (x, y, y)) :
- $t \leftarrow$ `flip` t
- $t \leftarrow$ `zagier` t
- end while.
- return t, which is a `flip` fix.

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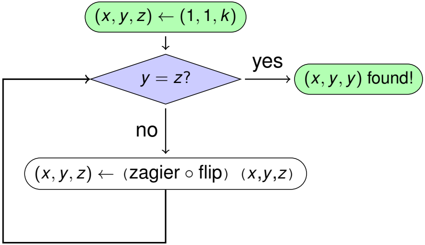

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- `flip` is just swapping y, z of triple.
- `zagier` has double, add and subtract only.
- Algorithm is simple, and fast for ✓ primes n not-too-large.
- Need to prove: code is correct, and will terminate for a ✓ prime.

Algorithm Run

Executing the algorithm in HOL4 for ✓ prime $n = 37$:

```
> EVAL ``two_squares 37``;
val it = |- two_squares 37 = (1,6): thm
```

HOL4 allows timing of an execution:

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> time EVAL ``two_squares 37``;
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```

More examples:

```
> time EVAL ‘‘two_squares 97‘‘;
runtime: 0.00770s, gctime: 0.00086s, systime: 0.00077s.
val it = |- two_squares 97 = (9,4): thm
> time EVAL ‘‘two_squares 1999999913‘‘;
runtime: 2m23s, gctime: 14.7s, systime: 11.3s.
val it = |- two_squares 1999999913 = (1093,44708): thm
> time EVAL ‘‘two_squares 12345678949‘‘;
runtime: 6m02s,    gctime: 37.5s,     systime: 26.0s.
val it = |- two_squares 12345678949 = (110415,12418): thm
```

Algorithm Correctness

Preprint of paper at arXiv: `https://arxiv.org/abs/2112.02556`

Windmills of the Minds: An Algorithm for Fermat's Two Squares Theorem

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Abstract

The two squares theorem of Fermat is a gem in number theory, with a spectacular one-sentence "proof from the Book". Here is a formalisation of this proof, with an interpretation using windmill patterns. The theory behind involves involutions on a finite set, especially the parity of the number of fixed points in the involutions. Starting as an existence proof that is non-constructive, there is an ingenious way to turn it into a constructive one. This gives an algorithm to compute the two squares by iterating the two involutions alternatively from a known fixed point.

CCS Concepts: • **Theory of computation → Automated reasoning**;

Keywords: Number Theory, Algorithm, Iteractive Theorem Proving.

The involution on the finite set

$$S = \{(x, y, z) \in \mathbb{N}^3 \mid n = x^2 + 4yz\}$$

defined by

$$(x,y,z) \longmapsto \begin{cases} (x + 2z, z, y - z - x) & \text{if } x < y - z \\ (2y - x, y, x + z - y) & \text{if } y - z < x < 2y \\ (x - 2y, x + z - y, y) & \text{if } x > 2y \end{cases} \quad (1)$$

has exactly one fixed point, so $|S|$ is odd, and the involution defined by $(x,y,z) \longmapsto (x,z,y)$ also has a fixed point.

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Orbits connecting fixed points.

Orbits with odd or even periods.

Thank you!